Assignment 9

The Cauchy problem for a single equation is given by

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1}$$

where $f \in C(R)$ satisfies the Lipschitz condition in some rectangle R and (x_0, y_0) belongs to the interior of R.

1. Show that the solution to (1) belongs to C^{k+1} (as long as it exists) provided $f \in C^k(R)$ for $k \geq 1$. In particular, $y \in C^\infty$ provided $f \in C^\infty(R)$.

2. In our proof of Picard-Lindelöf theorem it was shown that the solution of (1) exists on $[x_0 - a', x_0 + a']$ where $0 < a' < \min\{a, b/M, 1/L^*\}$. Prove that in fact the solution exists in $0 < a' < \min\{a, b/M\}$, that is, the Lipschitz condition is not involved.

3. Let $f \in C^1(G)$ where G is open in $\mathbb{R}^2$. Show that f satisfies the Lipschitz condition on every compact subset of G. Suggestion: Argue by contradiction. If not, $\exists (y_n, z_n) \to (y_0, z_0) \in K$ such that $|f(x, y_n) - f(x, z_n)| \geq n|y_n - z_n|$ so $y_0 = z_0$, etc.

4. Find the maximal interval of existence for the following Cauchy problem. Specify G first.

   (a)
   $$f_1(x, y) = \frac{1}{xy}, \quad y(1) = 1,$$

   (b)
   $$f_2(x, y) = y + e^x \sin x, \quad y(0) = -2,$$

   (c)
   $$f_3(x, y) = y^a \quad (0 < a < 1, a > 1), \quad y(1) = 1,$$

   (d)
   $$f_4(x, y) = \sin \frac{1}{1 - y + x}, \quad x_0 = y_0 = 0.$$

5. Consider the Cauchy problem for $f(x, y) = \alpha y(M - y)$, $\alpha, M > 0$.

   (a) Find the maximal interval of solution corresponding to the initial data $y(0) = a$ as a varies over $(-\infty, \infty)$.

   (b) In this logistic model $y(x)$ gives the population of some species at time x. Show that $y(x) \to M$ whenever $y(0) > 0$. In other words, $y(x) \equiv M$ is a stable equilibrium state for this model and the other steady state $y(x) \equiv 0$ is unstable.

6. A comparison principle. Let y_1 and y_2 satisfy the differential inequalities

   $$y_1' \leq f_1(x, y_1), \qquad \text{and} \qquad y_2' \geq f_2(x, y_2),$$

   respectively with initial data $y_i(x_0) = y_{0i}$, $i = 1, 2$. Show that $y_1(x) < y_2(x)$, $x \geq x_0$, as long as they exist provided $y_{01} < y_{02}$, $f_1(x, y) \leq f_2(x, y)$ for all x, y and $f_2(\cdot, y)$ is strictly increasing in y. Here $f_1, f_2 \in C(R)$.

7. (a) Show that the Cauchy problem

   $$y' = 1 + |y|^\gamma, \ y(x_0) = y_0, \ \gamma > 1,$$

   cannot have a global solution, that is, a solution in $\mathbb{R}$.

(b) Show that the Cauchy problem

$$y' = g(x, y), \ y(x_0) = y_0 > 0,$$

where $g \in C^1(\mathbb{R}^2)$ has a local but not a global solution if

$$g(x, y) \geq y^\gamma, \quad \forall y > 0,$$

where $\gamma > 1$.

8. Let $f \in C(\mathbb{R}^2)$ which satisfies the Lipschitz condition on every compact rectangle and

$$|f(x, y)| \leq C(1 + |y|), \quad (x, y) \in \mathbb{R}^2,$$

for some constant C. Show that (1) admits a global solution in $(-\infty, \infty)$. Hint: Use comparison principle.

9. Optional. Continuous dependence on initial data. We may consider the unique solution y as a function of both x and y_0 while x_0 remains fixed.

(a) Show that the map $y_0 \mapsto y(x, y_0)$ is continuous for fixed x.

(b) Show that further when $f \in C^1(R)$, this map is continuously differential near y_0 for fixed x. Hint: Let z be the solution to the linear Cauchy problem

$$z' = \frac{\partial f}{\partial y}(x, y(x, y_0))z, \quad z(x_0) = 1,$$

where $y(x, y_0)$ denotes the solution of (1). Show that

$$\lim_{h \to 0} \frac{y(x, y_0 + h) - y(x, y_0)}{h} = z(x).$$

Use the fact that the function $q(x) \equiv \dfrac{y(x, y_0 + h) - y(x, y_0)}{h}$ satisfies a linear equation of the form

$$q' = \frac{\partial f}{\partial y}(x, y(x, y_0))q + b(x),$$

where b is small in some sense.

10. Show that there exists a unique solution h to the integral equation

$$h(x) = 1 + \frac{1}{\pi} \int_{-1}^{1} \frac{1}{1 + (x - y)^2} h(y) dy,$$

in $C[-1, 1]$. Also show that h is non-negative.